Converting levels to rates of change

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1 Basic percentage changes

It's often important to calculate percentage changes in economics. Here is a reminder of what a percentage change is and then a few quick and easy methods to figure out the percentage change of something.

The easiest way to think about the percentage change of any variable is to think "new minus old over old". The formula is this:

$$\%\Delta X = \left(\frac{X_n - X_o}{X_o}\right) \times 100$$

This says "New X (X_n) minus old X (X_o) divided by old X(X_o) times 100". You can even drop the 100 at the end. But it depends on what you want. Technically it's there, hence the word "percent".

Another useful formulation - which I use mostly in excel - is to rewrite it like this:

$$\%\Delta X = \left(\frac{X_n}{X_o} - 1\right) \times 100$$

And, finally, conceptually, to emphasize the percentage *change* I write it this way:

$$\%\Delta X = \left(\frac{\Delta X}{X_o}\right) \times 100$$

This reminds us that it is the change relative to the starting point, hence "percentage change".

1.1 Basic converting to percentage changes 1: small changes

An undergrad professor of mine once called this "hat algebra" and the name stuck with me. For my undergraduate students I think it's less intimidating and totally different so it doesn't sound too "mathy".

Let's start with just putting the two rules out there:

$$XY \rightarrow \%\Delta X + \%\Delta X$$

or, using little "hats" instead of $\%\Delta$.

$$\textbf{Rule 1: } XY \rightarrow \hat{X} + \hat{Y}$$

and

$$\textbf{Rule 2: } \frac{X}{Y} \rightarrow \hat{X} - \hat{Y}$$

In words **Rule One** says that if two variables multiply each other, then to figure out how they grow over time (i.e., when you convert them to percentage changes) add the percentage change to each. **Rule Two** says that if two variables divide, then their growth rates subtract.

1.1.1 A Few Applications

In economics we are often interested in, say, Nominal GDP (Y) or Nominal GDP per capita (Y/pop) but we're also interested in Nominal GDP growth or Nominal GDP growth per capita.

Let's start with Nominal GDP. Nominal GDP is the "market value of goods and services produced in an economy during a period of time (like a year)". To get market value we need to multiply the quantities by prices. At the aggregate level this means multiplying real GDP, y, but the price level, P, so $Y = Py$. To see how nominal GDP, Y, grows over time, convert it to "hats":

$$Y = Py \rightarrow \hat{Y} = \hat{P} + \hat{y}$$

This makes sense. Nominal GDP is the price level times real GDP. So it's growth rate is determined by how the price level *and* real GDP grow over time.

GPP per capita, Y/pop gives us a chance to apply the other rule.

$$\text{GDP per capita} = \frac{Y}{pop} \rightarrow \hat{Y} - \hat{pop}$$

So, GDP per capita is rising if GDP is rising faster than the population but falling if the population grows faster. Again, pretty logical.

There's nothing more to the basic application of "converting things to hats".

1.2 Basic converting to percentage changes 2: Handling large changes

The only warning so far is that this simple method is only accurate for small changes. If you try to calculate these changes in, say, excel, you'll find that this method is fairly accurate unless you do examples of big percentage changes. And the bigger the change, the less accurate. The reason (explained below with calculus) is that this method is based on extremely small changes (infinitesimally small) and can be thought of as an approximation method.

So I use the above method in class all the time because it gets the idea across and it's clean and simple. But, it's not accurate and when you need to calculate something you need to use the correct formulas which are here:

$$\textbf{Rule 1.B: } XY \rightarrow \hat{X} + \hat{Y} + \hat{X}\hat{Y}$$

and

$$\textbf{Rule 2.B: } \frac{X}{Y} \rightarrow \frac{\hat{X} - \hat{Y}}{1 + \hat{Y}}$$

These aren't that complicated. In teaching, I find that I just need to show the students these formulas when they calculate things in excel but I don't use them often in class. Usually it's enough to use the small change approximation and remind people "if you plug in actual numbers, it'll be off a little bit".

1.3 Basic converting to percentage changes 3: Handling exponents

Finally, it often happens that we need to convert to percentage changes something with exponents in the formula. The most common case in macroeconomics is to convert a Cobb Douglas production function to percentage change. The example below does this for "small changes". Here are the rules (note that they are both really Rule 1):

$$\textbf{Exponent Rule 1: } Z = X^{\alpha}Y^{\beta} \rightarrow \hat{Z} = \alpha\hat{X} + \beta\hat{Y}$$

$$\textbf{Exponent Rule 2: } Z = \frac{X^\alpha}{Y^\beta} \rightarrow \hat{Z} = \alpha\hat{X} - \beta\hat{Y}$$

Applying this to the Cobb Douglas production function, $y = AK^\alpha L^{1-\alpha}$ is a straight forward application of Rule 1:

$$y = AK^\alpha L^{1-\alpha} \rightarrow \hat{y} = \alpha\hat{K} + (1-\alpha)\hat{L}$$

2 Looking under the hood: The math behind the conversions

This section is aimed at the person who either has more math background or plans to go further in their studies and needs to understand the math actually involved.

2.1 The large and discrete change derivation

The accurate correct derivation uses discrete changes in the variables. It's easiest to think of change over time, so consider each variable as being a function of time.

$$Z = Z(t), X = X(t), Y = Y(t)$$

with associated growth rates, $\hat{Z}, \hat{X}, \hat{Y}$ so that $Z_{t+1} = (1+\hat{Z})Z_t$, $X_{t+1} = (1+\hat{X})X_t$, and $Y_{t+1} = (1+\hat{Y})Y_t$. Or, using our "new" and "old" designations,

$$Z_n = (1+\hat{Z})Z_o$$
$$X_n = (1+\hat{X})X_o$$
$$Y_n = (1+\hat{Y})Y_o$$

where new is old plus 1. You can think of this as the old date plus 1 day, or 1 week or 1 year. It's only important that it's 1 unit of time.

Now we are in a position to derive the first rule:

$$\textbf{Rule 1.B: } Z = XY \rightarrow \hat{Z} = \hat{X} + \hat{Y} + \hat{X}\hat{Y}$$

$$Z(t) = X(t)Y(t)$$
$$Z_n = X_nY_n$$
$$(1+\hat{Z})Z_o = (1+\hat{X})X_o(1+\hat{Y})Y_o$$
$$(1+\hat{Z}) = (1+\hat{X})(1+\hat{Y})$$
$$1+\hat{Z} = 1+\hat{X}+\hat{Y}+\hat{X}\hat{Y}$$

$$\hat{Z} = \hat{X}+\hat{Y}+\hat{X}\hat{Y}$$

The second rule is derived as follows:

$$\textbf{Rule 2.B: } Z = \frac{X}{Y} \rightarrow \hat{Z} = \frac{\hat{X}-\hat{Y}}{1+\hat{Y}}$$

$$Z(t) = \frac{X(t)}{Y(t)}$$

$$Z_n = \frac{X_n}{Y_n}$$

$$(1+\hat{Z})Z_o = \frac{(1+\hat{X})X_o}{(1+\hat{Y})Y_o}$$
$$(1+\hat{Z}) = \frac{(1+\hat{X})}{(1+\hat{Y})}$$
$$(1+\hat{Z})(1+\hat{Y}) = (1+\hat{X})$$
$$1+\hat{Z}+\hat{Y}+\hat{Z}\hat{Y} = 1+\hat{X}$$
$$\hat{Z}+\hat{Z}\hat{Y} = \hat{X}-\hat{Y}$$
$$\hat{Z}(1+\hat{Y}) = \hat{X}-\hat{Y}$$

$$\hat{Z} = \frac{\hat{X}-\hat{Y}}{1+\hat{Y}}$$

2.2 Approximation

The reason the approximation (i.e., the basic hat algebra) works is because for small enough changes, the $\hat{X}\hat{Y}$ piece basically goes to zero. To see this, let each change be 50%, so $\hat{X} = .5$ and $\hat{Y} = .5$. Then,$\hat{X}\hat{Y} = (.5)(.5) = .25$. But as each percentage change gets smaller, say, 5% or .05, this multiplied term gets much smaller and very fast, now $(.05)(.05) = .0025$. And therefore as each percentage gets small, this multiplicative term goes to zero.

As a result, you can see in the derivations of each expression where letting this $\hat{X}\hat{Y}$ term go to zero gets you to the original formulas.

$$\textbf{Rule 1.B: } Z = XY \rightarrow \hat{Z} = \hat{X}+\hat{Y}+\underbrace{\hat{X}\hat{Y}}_{=0}$$

so, we just get $\hat{Z} = \hat{X}+\hat{Y}$ which is the original, simple formula.

It's not as obvious for Rule 2, but in the derivation you can see it two lines from the end,

$$\hat{Z}+\underbrace{\hat{Z}\hat{Y}}_{=0} = \hat{X}-\hat{Y}$$

so, we just get $\hat{Z} = \hat{X}-\hat{Y}$ which is the original, simple formula, "Rule 2".

3 Derivation with calculus: appropriate for small changes

Last but not least is to derive these relationships with calculus. A first method would be along the lines discussed above. Use the discrete change and then take the limit as the changes get infinitely small. But that approach doesn't make handling exponents obvious.

The calculus approach is basically a two step proceedure:

Step 1: take natural logs.
Step 2: take a derivative with respect to time.

Recall, $Z = Z(t)$, $X = X(t)$, and $Y = Y(t)$. To derive the first rule,

$$\textbf{Rule 1: } Z = XY \rightarrow \hat{Z} = \hat{X}+\hat{Y}$$

$$Z(t) = X(t)Y(t)$$

$$
\begin{aligned}
ln(Z) &= ln(X) + ln(Y) \\
\frac{dln(Z)}{dt} &= \frac{dln(X)}{dt} + \frac{dln(Y)}{dt} \\
\frac{dZ}{dt}\frac{1}{Z} &= \frac{dX}{dt}\frac{1}{X} + \frac{dY}{dt}\frac{1}{Y}
\end{aligned}
$$

Let $\frac{dZ}{dt} = \dot{Z}$ represent an infinitesimally small change. That is, $\dot{Z}$ is an infinitesimally small ΔZ. Recall that we pointed out that a percentage change is just a change relative to the old value. Well, that's what we have here. Rewriting this last expression:

$$
\frac{\dot{Z}}{Z} = \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}
$$

$$
\hat{Z} = \hat{X} + \hat{Y}
$$

This method makes exponents easy to handle since "logs" just "bring the exponent down". So,

$$
\begin{aligned}
Z(t) &= X(t)^{\alpha} Y(t)^{\beta} \\
ln(Z) &= \alpha ln(X) + \beta ln(Y) \\
\frac{dln(Z)}{dt} &= \alpha \frac{dln(X)}{dt} + \beta \frac{dln(Y)}{dt}
\end{aligned}
$$

$$
\frac{\dot{Z}}{Z} = \alpha \frac{\dot{X}}{X} + \beta \frac{\dot{Y}}{Y}
$$

$$
\hat{Z} = \alpha \hat{X} + \beta \hat{Y}
$$

Finally, to handle the case of division, $Z = X/Y$, one can just use exponents and set $\alpha = 1$ and $\beta = -1$ to get the second basic rule we started with, $\hat{Z} = \hat{X} - \hat{Y}$.